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Geo/G/1 queues with disasters and general repair times

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A B S T R A C T

This paper discusses discrete-time single server Geo/G/1 queues that are subject to failure due to a disaster arrival. Upon a disaster arrival, all present customers leave the system. At a failure epoch, the server is turned off and the repair period immediately begins. The repair times are commonly distributed random variables. We derive the probability generating functions of the queue length distribution and the FCFS sojourn time distribution. Finally, some numerical examples are given.

1. Introduction

For several decades, queueing systems with negative arrivals (of mainly negative customers or disasters) have garnered increasing attention due to their applications to computer networks, communication systems and manufacturing systems.

If a negative customer arrives at a queueing system, it forces a customer in service to leave the system. First introduced by Gelenbe [1], the concept of negative customers has been applied to the M/M/1 queue [2], the M/G/1 queue [3], and the GI/M/1 queue [4]. Excellent surveys on negative customers are provided by Artalejo [5] and Gelenbe [6,7].

On the other hand, if a disaster arrives at a system, it removes all present customers (i.e., a customer in service plus customers in queue) in the system at once. Disasters are also referred to as 'mass exodus' [8], 'queue flushing' [9], 'catastrophes' [10,11], and 'stochastic clearing systems' [12,13]. Disasters can be regarded as the breakdown of a machine and the resulting destruction of all work in process in manufacturing systems. In computer networks and telecommunication systems, if a file is infected by a virus, this infected file may transmit the virus when it is transferred to other processors such as CPU, I/O devices, diskettes, etc. Therefore, we may think of a virus infection as a disaster that destroys all stored files. Recently, Yang et al. [14] applied disasters including negative customers to the evaluation of information security investment with security threats such as computer viruses.

Since the notion of disasters was introduced by Towsley and Tripathi [9], a number of significant studies on queues with disasters have been published. In [9], the authors studied the M/M/1 queue with disasters to analyze a distributed database system with site failure. This M/M/1 queue was extended to the M/G/1 queue by Jain and Sigman [15]. Yang et al. [13] also analyzed the M/G/1 queue with disasters where a repairable server was assumed. Recently, Economou and Kapodistria [16], Yechiali [17], and Sudhesh [18] considered the M/M/1 queues with disasters where customers are impatient due to the

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absence of the server(s). The results of the queueing system with disasters and impatient customers under the assumption of a Markovian arrival process (MAP) can be found in Chakravarthy [19]. Gómez-Corral [20] dealt with a finite-buffer bulk-service queue with disasters where the arrival streams of customers and disasters are MAPs.

This topic was recently extended to a discrete-time queue with negative customers and disasters. It should be noted that a discrete-time queue is more suitable for describing the operation of time-slotted digital communication systems including mobile and broad integrated services digital networks (B-ISDN) based on asynchronous transfer mode (ATM) technology due to the packetized nature of these transport protocols. Atencia and Moreno [21] studied the Geo/Geo/1 queues with negative customers and various killing strategies caused by the negative customers. Wang and Zhang [22] considered the Geo/G/1 retrial queue with negative customers and an unreliable server. Recently, the GI/Geo/1 queues with negative customers and various killing strategies were analyzed by Chae et al. [23].

While many continuous-time queueing systems with disasters have been studied, their discrete-time counterparts have received very little attention in the literature. Atencia and Moreno [24] presented the results of the Geo/Geo/1 queue with disasters in which it was assumed that each arrival has a certain probability of being a customer who will be served or of being a disaster. However, it is more reasonable to assume that a disaster arrival is independent of a customer arrival, as done in this paper. Jolai et al. [25] modeled the operation of an email contact center as the finite capacity Geo/Geo/1/∞/N queue with disasters. Yi et al. [26] studied the Geo/G/1 queue with disasters to analyze the Geo/G/1 queue with multiple working vacations. Recently, Park et al. [27] extended the Geo/Geo/1 queue with disasters to the GI/Geo/1 queue with disasters, where the interarrival times are generally distributed. In this paper, we consider the Geo/G/1 queues with disasters under the assumption of a repairable server. It is realistic to consider a repair since a disaster represents a server breakdown that destroys all works in a system. Our study extends the work of Yang et al. [13] in a discrete-time system.

We consider the following features for our models. Customers arrive at a single server queue, according to a Bernoulli process. A server provides a service to each customer on a FCFS basis. The service times are independent and identically distributed (i.i.d.) random variables commonly distributed. Disasters arrive when the server is busy, according to a Bernoulli process. (In other words, disasters do not arrive when the server is idle or under repair.) Each time disasters arrive at a system, the server fails and all present customers leave the system simultaneously. At a failure epoch, the server is turned off and a repair period immediately begins. The server repair times are i.i.d. random variables commonly distributed. During the repair period, the stream of new arrivals continues. In System 1, these customers newly arriving during the repair period cannot enter the system and are blocked. On the other hand, in System 2, they join the queue and wait for the server to be repaired. As soon as the repair period ends, the server promptly becomes available. For each system, using the probability generating function (PGF) technique, we present PGFs of the queue length distribution and the sojourn time distribution.

The remainder of this paper is organized as follows. In Section 2, the mathematical model under consideration is described. In Sections 3 and 4, general results on the queue length and the sojourn time of System 1 and System 2 are presented. Numerical experiments are conducted to investigate the influence of the arrival of disasters on the mean queue length of each system.

2. Model description

We adopt the late arrival system (LAS) [28]. Let the time axis be marked by $t = 0,1,2,\ldots$ According to the LAS model, a potential customer arrival occurs during the interval (t^-,t) and a potential service completion occurs during the interval (t,t^+), where t^+ and t^- represent $\lim_{\Delta t\to 0}(t + |\Delta t|)$ and $\lim_{\Delta t\to 0}(t - |\Delta t|)$, respectively. Since a customer arrival and a disaster arrival can occur simultaneously at a slot boundary, the order of these events must be stated. We assume that a potential disaster arrival occurs during the interval (t^-,t) and immediately before a potential customer arrival. Concerning the order of a customer arrival and a repair completion, we make the same assumption as the disaster arrival. We further assume that a disaster arrival and a repair completion do not occur at the same slot boundary simultaneously.

We define commonly used notations to analyze both System 1 and System 2. Interarrival times of customers $\{A_n\}_{n=1}^{\infty}$ are i.i.d. discrete random variables and follow a geometric distribution:

$$\Pr\{A_n = k\} = \bar{\lambda}^{k-1}\lambda, \quad k \geqslant 1; \quad \bar{\lambda} = 1-\lambda, \quad 0<\lambda<1; \quad E[A_n] = \lambda^{-1}.$$

Interarrival times of disasters $\{D_n\}_{n=1}^{\infty}$ are i.i.d. discrete random variables and follow a geometric distribution:

$$\Pr\{D_n = k\} = \bar{\delta}^{k-1}\delta, \quad k \geqslant 1; \quad \bar{\delta} = 1-\delta, \quad 0<\delta<1; \quad E[D_n] = \delta^{-1}.$$

Service times $\{S_n\}_{n=1}^{\infty}$ are i.i.d. discrete random variables and have the following distribution:

$$\Pr\{S_n = k\} = s_k, \quad k \geqslant 1; \quad S(z) = \sum_{k=1}^{\infty} s_k z^k.$$

Repair times $\{R_n\}_{n=1}^{\infty}$ are i.i.d. discrete random variables and have the following distribution:

$$\Pr\{R_n = k\} = r_k, \quad k \geqslant 1; \quad R(z) = \sum_{k=1}^{\infty} r_k z^k.$$

We assume that $\{A_n\}_{n=1}^{\infty}$, $\{D_n\}_{n=1}^{\infty}$, $\{S_n\}_{n=1}^{\infty}$, and $\{R_n\}_{n=1}^{\infty}$ are mutually independent.

System 1 and System 2 are represented by a Markov chain. Let $N(t^+)$ be the number of customers in the system at t^+. Let $\xi(t^+)$ be the server state at t^+ and defined as follows:

$$\xi(t^+) = \begin{cases} 0, & \text{The server is under repair at } t^+, \\ 1, & \text{The server is available at } t^+. \end{cases}$$

$\{N(t^+), \xi(t^+), S_R(t^+), R_R(t^+), t = 0, 1, \ldots\}$ is then the Markov chain, where the supplementary variables $S_R(t^+)$ and $R_R(t^+)$ respectively denote the remaining service time and the remaining repair time all at t^+.

3. System 1

In System 1, newly arriving customers are blocked when the server is under repair.

3.1. Queue length distribution

Let us define the following limiting probabilities:

$$\tilde{\pi}_0(k) = \lim_{t\to\infty} \Pr\{N(t^+) = 0, \xi(t^+) = 0, R_R(t^+) = k\},$$
$$\pi_n(k) = \lim_{t\to\infty} \Pr\{N(t^+) = n, \xi(t^+) = 1, S_R(t^+) = k\}, \quad n \geqslant 1,$$
$$\tilde{\pi}_0 = \lim_{t\to\infty} \Pr\{N(t^+) = 0, \xi(t^+) = 0\} = \sum_{k=1}^{\infty} \tilde{\pi}_0(k),$$
$$\pi_0 = \lim_{t\to\infty} \Pr\{N(t^+) = 0, \xi(t^+) = 1\},$$
$$\pi_n = \lim_{t\to\infty} \Pr\{N(t^+) = n, \xi(t^+) = 1\} = \sum_{k=1}^{\infty} \pi_n(k), \quad n \geqslant 1.$$

Considering mutually exclusive events that can occur during one slot, we have following Kolmogorov equations.

$$\tilde{\pi}_0(k) = \tilde{\pi}_0(k+1) + \delta r_k \sum_{n=1}^{\infty} \pi_n, \tag{3.1}$$
$$\pi_0 = (\tilde{\pi}_0(1) + \pi_0 + \bar{\delta}\pi_1(1))\bar{\lambda}, \tag{3.2}$$
$$\pi_1(k) = \pi_0 \lambda s_k + \bar{\delta}(\pi_1(1)\lambda s_k + \pi_1(k+1)\bar{\lambda} + \pi_2(1)\bar{\lambda} s_k) + \tilde{\pi}_0(1)\lambda s_k, \tag{3.3}$$
$$\pi_n(k) = \bar{\delta}(\pi_{n-1}(k+1)\lambda + \pi_n(1)\lambda s_k + \pi_n(k+1)\bar{\lambda} + \pi_{n+1}(1)\bar{\lambda} s_k), \quad n \geqslant 2. \tag{3.4}$$

First, we sum both sides of (3.1) over k, $k \geqslant 1$. We also multiply both sides of (3.1) by k and sum over k, $k \geqslant 1$. In other words,

$$\sum_{k=1}^{\infty} \tilde{\pi}_0(k) = \sum_{k=1}^{\infty} \tilde{\pi}_0(k+1) + \delta \sum_{k=1}^{\infty} r_k \sum_{n=1}^{\infty} \pi_n,$$
$$\sum_{k=1}^{\infty} k\tilde{\pi}_0(k) = \sum_{k=1}^{\infty} k\tilde{\pi}_0(k+1) + \delta \sum_{k=1}^{\infty} k r_k \sum_{n=1}^{\infty} \pi_n.$$

Simplifying the above equations results in

$$\tilde{\pi}_0(1) = \delta \sum_{n=1}^{\infty} \pi_n, \tag{3.5.a}$$
$$\tilde{\pi}_0 = \tilde{\pi}_0(1) E[R], \tag{3.5.b}$$

which implies that $\tilde{\pi}_0 = \delta E[R] \sum_{n=1}^{\infty} \pi_n$.

In order to solve the Kolmogorov equations, let us define the following PGFs:

$$\Pi(z, k) = \sum_{n=1}^{\infty} \pi_n(k) z^n, \quad |z| \leqslant 1,$$
$$\Pi^*(z, w) = \sum_{k=1}^{\infty} \Pi(z, k) w^k, \quad |w| \leqslant 1.$$

Note that $\Pi^*(z, 1) = \sum_{n=1}^{\infty} \pi_n z^n$. In this paper, we define P_I, P_B, and P_R as the probability that the server is idle, busy, and under repair, respectively. It is then easy to verify that $P_B = \Pi^*(1, 1) = \sum_{n=1}^{\infty} \pi_n$ and $P_R = \sum_{k=0}^{\infty} \tilde{\pi}_0(k) = \tilde{\pi}_0$. Consequently, P_I is equal to π_0. Taking into consideration the foregoing, the normalizing condition is given by

$$\pi_0 + \tilde{\pi}_0 + \Pi^*(1, 1) = 1. \tag{3.6}$$

Multiplying (3.3) and (3.4) by z^n and summing over n, $n \geqslant 1$, we obtain

$$\Pi(z,k) = \omega_0 \Pi(z,k+1) + s_k\left[z^{-1}\omega_0\Pi(z,1) + \tilde{\pi}_0(1)(\bar{\lambda} + \lambda z) - \pi_0\lambda(1-z)\right], \tag{3.7}$$

where $\omega_0 = \bar{\delta}(\bar{\lambda} + \lambda z)$. Multiplying (3.7) by w^k and summing over k, $k \geqslant 1$, it finally yields

$$\Pi^*(z,w)(1 - w^{-1}\omega_0) = z^{-1}\omega_0\Pi(z,1)(S(w) - z) + S(w)[\tilde{\pi}_0(1)(\bar{\lambda} + \lambda z) - \pi_0\lambda(1-z)]. \tag{3.8}$$

Inserting $w = \omega_0$ in (3.8) and solving $\Pi(z,1)$, we obtain

$$\Pi(z,1) = \frac{zS(\omega_0)[\pi_0\lambda(1-z) - \tilde{\pi}_0(1)(\bar{\lambda} + \lambda z)]}{\omega_0(S(\omega_0) - z)}. \tag{3.9}$$

Substituting (3.5) and (3.9) into (3.8) gives

$$\Pi^*(z,w) = \frac{zw(S(w) - S(\omega_0))[\pi_0\lambda(1-z) - \delta(\bar{\lambda} + \lambda z)P_B]}{(w - \omega_0)(S(\omega_0) - z)}. \tag{3.10}$$

Let $P(z)$ denote the PGF of the queue length in the system. Letting $w = 1$ in (3.10), we obtain

$$P(z) = \pi_0 + \delta E[R]P_B + \frac{z(1 - S(\omega_0))[\pi_0\lambda(1-z) - \delta(\bar{\lambda} + \lambda z)P_B]}{(1 - \omega_0)(S(\omega_0) - z)}. \tag{3.11}$$

Rouche's theorem confirms that $S(\omega_0) - z = 0$ has a unique solution, say z^*, within a unit circle $|z| < 1$ (see Appendix A). If a denominator of $\Pi^*(z,1)$ is zero when $z = z^*$, the numerator should be zero under the same condition of $z = z^*$. Exploiting, we first obtain

$$\pi_0 = \frac{\delta(\bar{\lambda} + \lambda z^*)}{\lambda(1 - z^*)}P_B. \tag{3.12}$$

Then, by the normalizing condition, we finally have

$$\pi_0 = \frac{\delta(\bar{\lambda} + \lambda z^*)}{\delta(\bar{\lambda} + \lambda z^*) + \lambda(1 + \delta E[R])(1 - z^*)}, \tag{3.13.a}$$

$$\tilde{\pi}_0 = \frac{\lambda\delta E[R](1 - z^*)}{\delta(\bar{\lambda} + \lambda z^*) + \lambda(1 + \delta E[R])(1 - z^*)}, \tag{3.13.b}$$

$$P_B = \frac{\lambda(1 - z^*)}{\delta(\bar{\lambda} + \lambda z^*) + \lambda(1 + \delta E[R])(1 - z^*)}. \tag{3.13.c}$$

Remark 1. The system is stable if and only if $\delta > 0$. π_0 has a positive value as long as $\delta > 0$. This condition is necessary for the stability. Next, (3.12) verifies that $\delta > 0$ if π_0 is positive. This is the sufficient condition for the stability.

Remark 2. Letting $\delta = 0$ in (3.11) leads to an identical form as $P(z)$ of the Geo/G/1 queue.

3.2. FCFS sojourn time distribution

In this section, we derive the PGF of the FCFS sojourn time (i.e., the waiting time plus the service time) of a test customer (TC), regardless of whether its service is interrupted by a disaster or not. We do not take blocked customers' sojourn times into consideration because they are equal to 0. In this paper, $X(z)$ denotes the PGF of any discrete random variable X.

Let W denote the FCFS sojourn time of a TC. Suppose that the TC arrives when the server is available (either idle or busy). Note that all customers arriving when the server is available enter the system to be served. Let U denote the unfinished work immediately after the TC's arrival. As Bernoulli Arrivals See Time Average (BASTA) [28], $U(z)$ is expressed as follow:

$$U(z) = \frac{1}{1 - \tilde{\pi}_0}\left(\pi_0 S(z) + \frac{\Pi^*(S(z), z)}{z}\right), \tag{3.14}$$

where $\Pi^*(z,w)$ is given in (3.10). The probability mass function (PMF) of W and $W(z)$ are then given by

$$\Pr\{W = k\} = \Pr\{U = k, D \geqslant k+1\} + \Pr\{U \geqslant k, D = k\} = \bar{\delta}^{k-1}[\bar{\delta}\Pr\{U = k\} + \delta\Pr\{U \geqslant k\}], \quad k \geqslant 1, \tag{3.15.a}$$

$$W(z) = \frac{\delta z + (1 - z)U(\bar{\delta}z)}{1 - \bar{\delta}z}. \tag{3.15.b}$$

4. System 2

In System 2, newly arriving customers enter the system while the server is under repair. They receive their service after repair.

4.1. Queue length distribution

Let us define the following limiting probabilities:

$$\tilde{\pi}_n(k) = \lim_{t\to\infty} \Pr\{N(t^+) = n, \xi(t^+) = 0, R_R(t^+) = k\}, \quad n \geqslant 0,$$

$$\pi_n(k) = \lim_{t\to\infty} \Pr\{N(t^+) = n, \xi(t^+) = 1, S_R(t^+) = k\}, \quad n \geqslant 1,$$

$$\tilde{\pi}_n = \lim_{t\to\infty} \Pr\{N(t^+) = n, \xi(t^+) = 0\} = \sum_{k=1}^{\infty} \tilde{\pi}_n(k), \quad n \geqslant 0,$$

$$\pi_0 = \lim_{t\to\infty} \Pr\{N(t^+) = 0, \xi(t^+) = 1\},$$

$$\pi_n = \lim_{t\to\infty} \Pr\{N(t^+) = n, \xi(t^+) = 1\} = \sum_{k=1}^{\infty} \pi_n(k), \quad n \geqslant 1.$$

Considering mutually exclusive events that can occur during one slot, we have the following Kolmogorov equations.

$$\tilde{\pi}_0(k) = \tilde{\pi}_0(k+1)\bar{\lambda} + \delta \sum_{n=1}^{\infty} \pi_n \bar{\lambda} r_k, \tag{4.1}$$

$$\tilde{\pi}_1(k) = \tilde{\pi}_0(k+1)\lambda + \tilde{\pi}_1(k+1)\bar{\lambda} + \delta \sum_{n=1}^{\infty} \pi_n \lambda r_k, \tag{4.2}$$

$$\tilde{\pi}_n(k) = \tilde{\pi}_{n-1}(k+1)\lambda + \tilde{\pi}_n(k+1)\bar{\lambda}, \quad n \geqslant 2, \tag{4.3}$$

$$\pi_0 = (\tilde{\pi}_0(1) + \pi_0 + \bar{\delta}\pi_1(1))\bar{\lambda}, \tag{4.4}$$

$$\pi_1(k) = \pi_0 \lambda s_k + \bar{\delta}(\pi_1(1)\lambda s_k + \pi_1(k+1)\bar{\lambda} + \pi_2(1)\bar{\lambda} s_k) + \tilde{\pi}_0(1)\lambda s_k + \tilde{\pi}_1(1)\bar{\lambda} s_k, \tag{4.5}$$

$$\pi_n(k) = \bar{\delta}(\pi_{n-1}(k+1)\lambda + \pi_n(1)\lambda s_k + \pi_n(k+1)\bar{\lambda} + \pi_{n+1}(1)\bar{\lambda} s_k) + \tilde{\pi}_{n-1}(1)\lambda s_k + \tilde{\pi}_n(1)\bar{\lambda} s_k, \quad n \geqslant 2. \tag{4.6}$$

It should be noted that, under our assumptions, the repair period may start with one customer existing in the system if a disaster arrival and a customer arrival occur simultaneously.

From (4.4), we obtain

$$\pi_0 \lambda = \tilde{\pi}_0(1)\bar{\lambda} + \bar{\delta}\pi_1(1)\bar{\lambda}. \tag{4.7}$$

In order to solve the Kolmogorov equations, let us define the following PGFs:

$$\tilde{\Pi}(z,k) = \sum_{n=0}^{\infty} \tilde{\pi}_n(k) z^n, \quad |z| \leqslant 1,$$

$$\Pi(z,k) = \sum_{n=1}^{\infty} \pi_n(k) z^n, \quad |z| \leqslant 1,$$

$$\tilde{\Pi}^*(z,w) = \sum_{k=1}^{\infty} \tilde{\Pi}(z,k) w^k, \quad |w| \leqslant 1,$$

$$\Pi^*(z,w) = \sum_{k=1}^{\infty} \Pi(z,k) w^k, \quad |w| \leqslant 1.$$

Let $P(z)$ denote the PGF of queue length distribution. Note that $\tilde{\Pi}^*(z,1) = \sum_{n=0}^{\infty} \tilde{\pi}_n z^n$ and $\Pi^*(z,1) = \sum_{n=1}^{\infty} \pi_n z^n$. Hence, $P(z)$ is represented as:

$$P(z) = \pi_0 + \tilde{\Pi}^*(z,1) + \Pi^*(z,1). \tag{4.8}$$

As a result, letting $z = 1$ in (4.8) yields the normalizing condition given by $\pi_0 + \tilde{\Pi}^*(1,1) + \Pi^*(1,1) = 1$. Like System 1, P_I and P_B are expressed as π_0 and $\Pi^*(1,1)$, respectively. Therefore, in System 2, P_R is expressed as $\tilde{\Pi}^*(1,1)$.

Multiplying (4.1)–(4.3) by z^n and summing over n, $n \geqslant 0$, we obtain

$$\tilde{\Pi}(z,k) = \omega_0(\delta r_k P_B + \tilde{\Pi}(z,k+1)), \tag{4.9}$$

where $\omega_0 = \bar{\lambda} + \lambda z$. Multiplying (4.8) by w^k and summing over k, $k \geqslant 1$, gives

$$\tilde{\Pi}^*(z,w)(1 - w^{-1}\omega_0) = \omega_0(\delta P_B R(w) - \tilde{\Pi}(z,1)). \tag{4.10}$$

Inserting $w = \omega_0$ in (4.10) and solving $\tilde{\Pi}(z,1)$, we obtain

$$\tilde{\Pi}(z,1) = \delta P_B R(\omega_0). \tag{4.11}$$

Substituting (4.11) into (4.10) yields

$$\tilde{\Pi}^*(z,w) = \frac{\delta P_B \omega_0 w(R(w) - R(\omega_0))}{w - \omega_0}. \tag{4.12}$$

Multiplying (4.5) and (4.6) by z^n and summing over n, $n \geqslant 1$, we obtain

$$\Pi(z,k) = \omega_1 \Pi(z,k+1) + s_k\Big(\omega_0 \tilde{\Pi}(z,1) + z^{-1}\omega_1 \Pi(z,1) + \pi_0 \lambda z - \pi_1(1)\bar{\lambda}\bar{\delta} - \tilde{\pi}_0(1)\bar{\lambda}\Big), \tag{4.13}$$

where $\omega_1 = \bar{\delta}(\bar{\lambda} + \lambda z)$. Multiplying (4.13) by w^k and summing over k, $k \geqslant 1$, finally yields

$$\Pi^*(z,w)(1 - w^{-1}\omega_1) = z^{-1}\omega_1 \Pi(z,1)(S(w) - z) + S(w)(\omega_0 \tilde{\Pi}(z,1) + \pi_0 \lambda z - \pi_1(1)\bar{\lambda}\bar{\delta} - \tilde{\pi}_0(1)\bar{\lambda}). \tag{4.14}$$

Inserting $w = \omega_1$ in (4.14) and solving $\Pi(z,1)$, we obtain

$$\Pi(z,1) = \frac{zS(\omega_1)(\pi_1(1)\bar{\lambda}\bar{\delta} + \tilde{\pi}_0(1)\bar{\lambda} - \pi_0 \lambda z - \omega_0 \tilde{\Pi}(z,1))}{\omega_1(S(\omega_1) - z)}. \tag{4.15}$$

Substituting (4.15) into (4.14) yields

$$\Pi^*(z,w) = \frac{zw\Big(\pi_1(1)\bar{\lambda}\bar{\delta} + \tilde{\pi}_0(1)\bar{\lambda} - \pi_0\lambda z - \omega_0\tilde{\Pi}(z,1)\Big)(S(w) - S(\omega_1))}{(w - \omega_1)(S(\omega_1) - z)}. \tag{4.16}$$

Using the results of (4.7) and (4.11), (4.16) is rewritten as follows:

$$\Pi^*(z,w) = \frac{zw(\pi_0\lambda(1 - z) - \delta P_B \omega_0 R(\omega_0))(S(w) - S(\omega_1))}{(w - \omega_1)(S(\omega_1) - z)}. \tag{4.17}$$

Letting $w = 1$ in both (4.12) and (4.17) and substituting the results into (4.8), we have

$$P(z) = \pi_0 + \frac{\delta P_B \omega_0 (1 - R(\omega_0))}{1 - \omega_0} + \frac{z(1 - S(\omega_1))(\pi_0 \lambda(1 - z) - \delta P_B \omega_0 R(\omega_0))}{(1 - \omega_1)(S(\omega_1) - z)}. \tag{4.18}$$

Similar to the analysis in Section 3, using Rouche's theorem, we obtain

$$\pi_0 = \frac{\delta(\bar{\lambda} + \lambda z^*)R(\bar{\lambda} + \lambda z^*)}{\lambda(1 - z^*)} P_B,$$

where z^* is a unique root of the equation $S(\omega_1) - z = 0$ within a unit circle $|z| < 1$. Next, substituting $z = 1$ into (4.12), it can be shown that $P_R = \delta E[R] P_B$. Then, by the normalizing condition, we determine unknowns as follows:

$$\pi_0 = \frac{\delta(\bar{\lambda} + \lambda z^*)R(\bar{\lambda} + \lambda z^*)}{\lambda(1 - z^*)(1 + \delta E[R]) + \delta(\bar{\lambda} + \lambda z^*)R(\bar{\lambda} + \lambda z^*)}, \tag{4.19.a}$$

$$P_R = \frac{\lambda \delta E[R](1 - z^*)}{\lambda(1 - z^*)(1 + \delta E[R]) + \delta(\bar{\lambda} + \lambda z^*)R(\bar{\lambda} + \lambda z^*)}, \tag{4.19.b}$$

$$P_B = \frac{\lambda(1 - z^*)}{\lambda(1 - z^*)(1 + \delta E[R]) + \delta(\bar{\lambda} + \lambda z^*)R(\bar{\lambda} + \lambda z^*)}. \tag{4.19.c}$$

Similar to Remark 1, System 2 is stable if and only if $\delta > 0$.

Remark 3. Letting $\delta = 0$ in (4.18) leads to an identical form as $P(z)$ of the Geo/G/1 queue.

4.2. FCFS sojourn time distribution

Let W denote the FCFS sojourn time of a TC, regardless of whether its service is interrupted by a disaster or not. We derive the $W(z)$ by using the conditional unfinished work.

Let us define π_A as the probability that the TC arrives when the server is available and π_R as the probability that TC arrives when the server is under repair. It is then clear that $\pi_A = P_I + P_B$ and $\pi_R = P_R$ due to BASTA.

First, we suppose that a TC arrives when the system is available. Let U_A denote the unfinished work immediately after the TC's arrival.

$$U_A(z) = \frac{1}{\pi_A}\left(\pi_0 S(z) + \frac{\Pi^*(S(z), z)}{z}\right). \tag{4.20}$$

Let W_A denote the sojourn time of the TC arriving when the system is available. The PMF of W_A and $W_A(z)$ are then given by

$$\Pr\{W_A = k\} = \Pr\{U_A = k, D \geqslant k+1\} + \Pr\{U_A \geqslant k, D = k\} = \bar{\delta}^{k-1}[\bar{\delta}\Pr\{U_A = k\} + \delta \Pr\{U_A \geqslant k\}], \quad k \geqslant 1, \tag{4.21.a}$$

$$W_A(z) = \frac{\delta z + (1 - z)U_A(\bar{\delta}z)}{1 - \bar{\delta}z}. \tag{4.21.b}$$

Now, we consider the sojourn time of the TC arriving during the repair time. Let U_R^n denote the unfinished work immediately after the arrival epoch of the TC who sees n customers during the repair time. Then, it is clear that $U_R^n(z) = [S(z)]^{n+1}$.

Let R_R denote the remaining repair time at the arrival epoch of the TC. Let R_R^n denote the remaining repair time at the arrival epoch of the TC who sees n customers in the system. We then have

$$R_R(z) = \frac{1}{\pi_R}\sum_{n=0}^{\infty}\tilde{\pi}_n R_R^n(z) = \frac{z(1-R(z))}{(1-z)E[R]}. \tag{4.22}$$

Let W_R^n denote the sojourn time of the arriving TC who sees n customers in the system during a repair time. W_R^n is represented as:

$$W_R^n = R_R^n + \min\left(U_R^n, D\right).$$

Since R_R^n is independent of both U_R^n and D, $W_R^n(z)$ is given by

$$W_R^n(z) = R_R^n(z)\cdot\frac{\delta z + (1-z)U_R^n(\bar{\delta}z)}{1-\bar{\delta}z}. \tag{4.23}$$

Let denote W_R the sojourn time of the TC arriving during the repair time. Applying BASTA, $W_R(z)$ is derived as follows:

$$W_R(z) = \sum_{n=0}^{\infty}\frac{\tilde{\pi}_n}{\pi_R}W_R^n(z) = \frac{\delta z}{1-\bar{\delta}z}\sum_{n=0}^{\infty}\frac{\tilde{\pi}_n}{\pi_R}R_R^n(z) + \frac{(1-z)S(\bar{\delta}z)}{1-\bar{\delta}z}\sum_{n=0}^{\infty}\frac{\tilde{\pi}_n}{\pi_R}R_R^n(z)[S(\bar{\delta}z)]^n. \tag{4.24}$$

Note that $\sum_{n=0}^{\infty}\tilde{\pi}_n z^n R_R^n(w)$ is the joint PGF of two variables: the queue length and the remaining repair time. Therefore, by using (4.12) and substituting (4.22) into (4.24), $W_R(z)$ can be simplified as

$$W_R(z) = \frac{\delta z}{1-\bar{\delta}z}\cdot\frac{z(1-R(z))}{(1-z)E[R]} + \frac{(1-z)S(\bar{\delta}z)}{1-\bar{\delta}z}\cdot\frac{\tilde{\Pi}^*(S(\bar{\delta}z),z)}{\pi_R}. \tag{4.25}$$

Finally, we have $W(z)$, given by

$$W(z) = \pi_A W_A(z) + \pi_R W_R(z). \tag{4.26}$$

5. Numerical examples

In this section, numerical examples are presented to show the influence of the arrival of disasters on the mean queue length of both System 1 and System 2. We also investigate the influence of the type of repair time distribution on the mean queue length. Let L_1 and L_2 respectively denote the mean queue lengths of System 1 and System 2. Then, differentiating each PGF of the queue length and letting $z = 1$ yields

$$L_1 = \frac{(1-\tilde{\pi}_0)\lambda}{\delta} - \frac{P_B S(\bar{\delta})}{1-S(\bar{\delta})}, \tag{5.1.a}$$

$$L_2 = \frac{\lambda}{\delta} + P_B\left(\frac{\lambda\delta E[R(R+1)]}{2} - \frac{S(\bar{\delta})}{1-S(\bar{\delta})}\right). \tag{5.1.b}$$

In all cases, customer arrivals are generated according to a Bernoulli process at rate 0.5. The service time distributions are assumed to follow one of the three distributions: a geometric distribution, a negative binomial distribution, and a mixture of two different geometric distributions. Specifically, for the geometric case, the PMF of the service time is defined by

$$s_k = \left(\frac{1}{8}\right)\left(\frac{7}{8}\right)^{k-1}, \quad k = 1, 2, \ldots.$$

For the negative binomial case, the PMF of the service time is defined by

$$s_k = \binom{k-1}{3}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{k-4}, \quad k = 4, 5, \ldots.$$

For the geometric mixture case, the PMF of the service time is defined by

$$s_k = \frac{4}{5}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{k-1} + \frac{1}{5}\left(\frac{1}{28}\right)\left(\frac{27}{28}\right)^{k-1}, \quad k = 1, 2, \ldots.$$

All three service times have a common mean of 8. The coefficients of variation are 0.94, 0.35, and 2 for the geometric, negative binomial, and geometric mixture case, respectively.

Similar to the service time, we consider three types of repair time distributions. For the geometric case, the PMF of the repair time is defined by

$$r_k = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{k-1}, \quad k = 1, 2, \ldots.$$

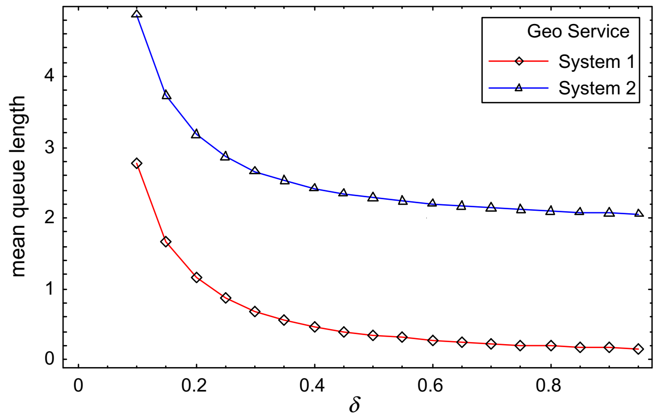

Fig. 1. Mean queue lengths versus δ in geometric service case.

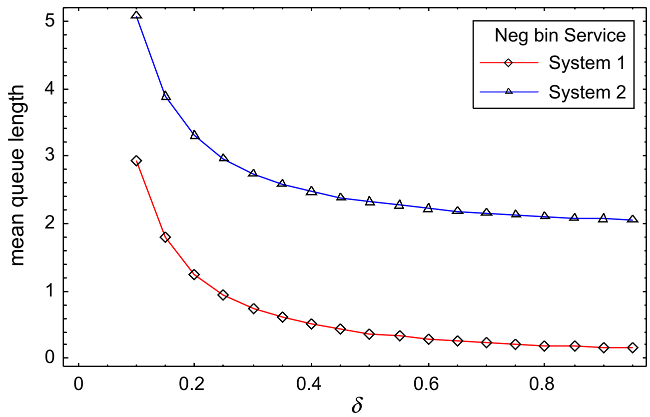

Fig. 2. Mean queue lengths versus δ in negative binomial service case.

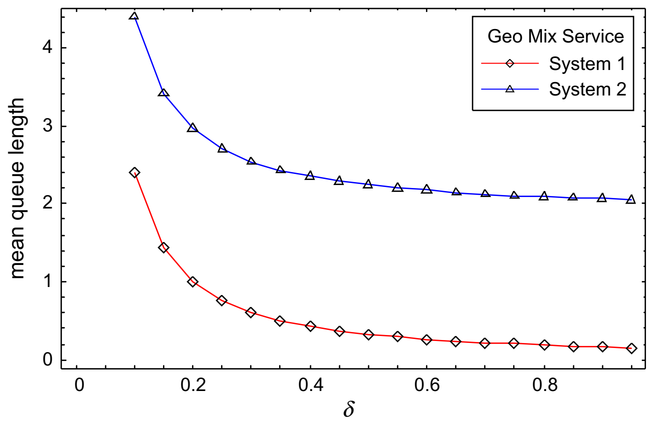

Fig. 3. Mean queue lengths versus δ in geometric mixture service case.

For the negative binomial case, the PMF of the repair time is defined by

$$r_k = \binom{k-1}{1}\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^{k-2}, \quad k = 2, 3, \ldots.$$

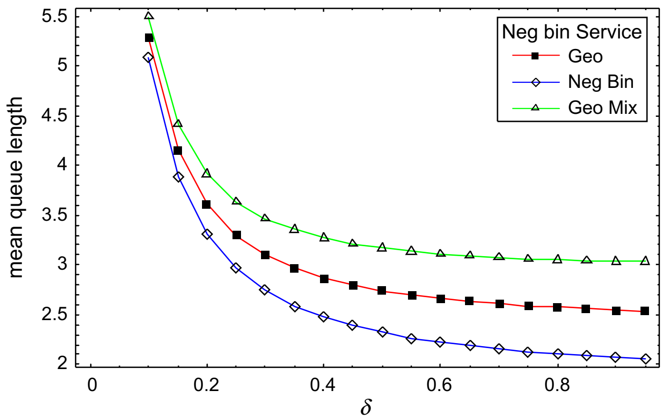

Fig. 4. Mean queue lengths versus δ with various repair time cases.

For the geometric mixture case, the PMF of the repair time is defined by

$$r_k = \frac{7}{10}\left(\frac{3}{20}\right)\left(\frac{17}{20}\right)^{k-1} + \frac{3}{10}\left(\frac{9}{10}\right)\left(\frac{1}{10}\right)^{k-1}, \quad k = 1, 2, \ldots .$$

While these three repair times have a common mean of 5, their coefficients of variation are 0.89, 0.55, and 1.15 for the geometric, negative binomial, and geometric mixture case, respectively.

In Figs. 1–3, L_1 and L_2 are shown as functions of δ when the repair time follows a negative binomial distribution. Figs. 1–3 confirm that both L_1 and L_2 decrease as δ increases. Furthermore, L_2 is greater than L_1 for all cases as expected. This is obvious because, in System 2, newly arriving customers during the repair period join the queue and wait for the server to be repaired without leaving the system.

Second, we investigate the tendencies of L_2 by varying the repair time distributions. In Fig. 4, L_2 is shown as a function of δ when the service time follows the negative binomial distribution.

As shown in Fig. 4, for all three different distributions of the repair times, as δ increases, the mean queue lengths decrease. Moreover, for all values of δ, the higher the coefficient of variation is, the greater the mean queue length is. The overall results confirm that the mean queue length is a decreasing function of δ for all cases.

Remark 4. As shown in (5.1.a), in System 1, the type of repair time distribution does not have an effect on L_1.

6. Concluding remarks

We considered Geo/G/1 queues with disasters and general repair times. At the instance of occurrence of disasters, all present customers leave the system and the server fails. At a failure epoch, the server is turned off and the repair period immediately begins. Those customers who arrive during the repair period are blocked in System 1, while remain in System 2. For each system, we presented the analytically explicit expressions for the PGFs of the queue length distribution and the FCFS sojourn time distribution. Finally, numerical works were carried out to survey the influence of the disasters' arrival rate on the systems. Our research enhances the practical value of disaster models in a discrete manner since it is realistic to consider the repair process. We anticipate that our results may help system architects in fields make better decisions on the repair policy.

It is noted that the results of the continuous-time M/G/1 queues with disasters and general repair times derived in [13] can be obtained from those derived in this paper by using Artalejo et al.'s technique introduced in Section 5 in [29]. As a result, continuous-time results could be obtained from those of the discrete-time results by substitution of matching terms (see Appendix B).

Appendix A. Application of Rouche's theorem

Let us define $S(\bar{\delta}(\bar{\lambda}+\lambda z))-z$ as $\phi(z)$, which is an analytic function in the unit circle $|z|<1$. Suppose $f(z)=-z$ and $g(z)=S(\bar{\delta}(\bar{\lambda}+\lambda z))$, which are all analytic. It can be shown that $|g(z)|<|f(z)|$ on the contour of the circle because

$$|f(z)| = |z| = 1,$$
$$|g(z)| \leqslant g(|z|) = S(\bar{\delta}(\bar{\lambda}+\lambda|z|)) = S(\bar{\delta}).$$

Hence, from Rouche's theorem, it follows that $f(z)$ and $f(z)+g(z)$ will have the same number of zeros inside $|z|<1$. Since $f(z)$ has only one zero inside this circle, $f(z)+g(z)\equiv\phi(z)$ will also have only one zero inside $|z|<1$.

Appendix B. A list of matching terms

A list of matching terms is as follows:

$$
\begin{aligned}
&S(z) \leftrightarrow S^*(\theta),\\
&S(\bar{\lambda} + \lambda z) \leftrightarrow S^*(\lambda - \lambda z),\\
&S(\bar{\delta}(\bar{\lambda} + \lambda z)) \leftrightarrow S^*(\delta + \lambda - \lambda z),\\
&R(z) \leftrightarrow R^*(\theta),\\
&R(\bar{\lambda} + \lambda z) \leftrightarrow R^*(\lambda - \lambda z),\\
&\frac{\delta z + (1-z)U(\bar{\delta}z)}{1 - \bar{\delta}z} \leftrightarrow \frac{\delta + \theta U^*(\theta+\delta)}{\theta+\delta},\\
&\frac{\delta z + (1-z)U_A(\bar{\delta}z)}{1 - \bar{\delta}z} \leftrightarrow \frac{\delta + \theta U_A^*(\theta+\delta)}{\theta+\delta},\\
&\frac{\delta z + (1-z)U_R^n(\bar{\delta}z)}{1 - \bar{\delta}z} \leftrightarrow \frac{\delta + \theta U_R^{n*}(\theta+\delta)}{\theta+\delta},
\end{aligned}
$$

where $X^*(\theta)$ is the Laplace Stieltjes transform of any continuous random variable X.

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